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Traveling waves and statistical distributions connected to systems of interacting populations

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ABSTRACT

We discuss the following two issues from the dynamics of interacting populations:

- (I) density waves for the case or negligible random fluctuations of the population densities,
- (II) probability distributions connected to the model equations for spatially averaged population densities for the case of significant random fluctuations of the independent quantity that can be associated with the population density.

For the case of issue (I) we consider model equations containing polynomial nonlinearities. Such nonlinearities arise as a consequence of interaction among the populations (for the case of large population densities) or as a result of a Taylor series expansion (for the case of small density of interacting populations). By means of the modified method of the simplest equation we obtain exact traveling-wave solutions of the model equations and these solution. For the case of issue (II) we discuss model equations of the Fokker–Planck kind for the evolution of the statistical distributions of population densities. We derive a few stationary distributions for the population density and calculate the expected exit time associated with the extinction of the studied population.

1. Introduction

Since the famous paper on the properties of the Lorenz attractor [1] nonlinearities have been intensively studied in different areas of science and especially in biology [2]. Just a few other examples are [3] from optics, [4] from biomechanics, [5] from solid state physics, [6,7] from fluid mechanics, [8,9] from time series analysis, etc. Population dynamics is a classic area of application of nonlinear models [10,11]. In many cases the dynamics of interacting populations is studied by mathematical models consisting of equations that contain only time dependence of the population densities [12–14]. These models are very useful for understanding the complex dynamics of the interacting populations but they do not account for two important aspects of this dynamics: (I) the possible influence of spatial characteristics of the environment; and (II) the possible fluctuations of the population densities caused by different factors. Below we shall investigate two kinds of population dynamics models that account for each of these effects. First of all we shall discuss the dynamics of spatially distributed populations and this will be a continuation of our previous work [15,16]. Then we shall show that by appropriate averaging the spatial model can be reduced to model in which the population densities depend only on the time. The models that contain spatially averaged quantities are valid for arbitrary values of the densities of the interacting populations. The result

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of the action of the fluctuations is that instead of equations for the trajectories of the populations in the phase space of the population densities we shall write and solve equations for the probability density functions of the population densities.

The organization of the paper is as follows. In Section 2 we introduce the model equations for a system of interacting spatially distributed populations and discuss their traveling wave solutions. In Section 3 we discuss the influence of fluctuations on the evolution of population densities of the model system of spatially averaged equations. The inclusion of random fluctuations transforms the model system of deterministic nonlinear ODEs to a system of Langevin equations which is further transformed to a system of Fokker–Planck equations for the evolution of the probability density functions (p.d.f.s) of interacting populations. We discuss a few examples for stationary p.d.f.s that are attractors for the time dependent p.d.f.s for the case of large times. A few expected exit times connected to extinction of populations are calculated too. A few concluding remarks are summarized in Section 4. In addition to the main text of the paper there are four appendices that are devoted to: (I) obtaining the model equations and their averaging; (II) the modified method of simplest equation used to obtain the traveling-wave solutions; (III) description of a MAPLE program for obtaining the analytic form of the solution for the coupled kink waves for the case of three interacting populations; and (IV) remarks on two observations connected to the diffusion Markov processes and the Fokker–Planck equation.

2. Model equations and traveling waves

In this paper we shall consider model systems of nonlinear PDEs for N interacting populations as follows (for more information about obtaining such models see Appendix A):

$$\frac{\partial \rho_i}{\partial t} - \sum_{k=1}^{N} D_{ik} \frac{\partial^2 \rho_k}{\partial x^2} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \dots \sum_{n_N=0}^{\infty} \alpha^{(i)}_{n_1,n_2,\dots,n_N} \rho_1^{n_1} \rho_2^{n_2} \dots \rho_N^{n_N}. \tag{2.1}$$

For the case of one population the system (2.1) is reduced to the equation

$$\frac{\partial \rho}{\partial t} - D \frac{\partial^2 \rho}{\partial x^2} = \sum_{n_1=0}^{\infty} \alpha_{n_1} \rho^{n_1}. \tag{2.2}$$

Below we shall discuss nonlinear PDEs of the kind (2.2) but with polynomial nonlinearity of finite order. We shall show how to obtain traveling wave kink solutions of such equations (if possible). For the lowest order of nonlinearity the corresponding PDEs are very famous (Fisher equation if the polynomial nonlinearity is of second order or Kolmogorov–Petrovskii–Piskunov equation if the polynomial nonlinearity is of third order). Many exact traveling wave solutions of these equations are known: see for an example the paper [17] for the Fisher equation or the paper of Ma and Fuchssteiner [18] for the Kolmogorov–Petrovskii–Piskunov equation. We note that the traveling wave kink solutions of the above class of equations are closely connected to some singularities in the complex plane that play a significant role in the test for the presence of the Painlevé property. An important part of this test is at the basis of the method of simplest equation (of which the version called the modified method of the simplest equation is discussed in Appendix B). We shall obtain the solutions of the studied nonlinear PDEs by the modified method of the simplest equation. Because of the connection of this method with the methodology of the Painlevé test the exact kink solutions obtained below are the most general kind of traveling wave single kink solutions of the corresponding nonlinear PDEs. All other single traveling wave kink solutions should be particular cases of the solutions obtained by the method of the simplest equation or should be the same. What will be important for us is to illustrate the methodology in the text below and not to pretend that all of the obtained solutions are new ones.

2.1. Traveling waves: case of one population

Let us discuss the simplest case of one population described by Eq. (2.2). First we introduce the traveling-wave coordinate $\xi = x - vt$ where v is the velocity of the wave. In addition we shall assume that the polynomial nonlinearity in Eq. (2.2) is up to order L. We rescale the coefficients in Eq. (2.2) as follows:

$$D^\dagger = -D/v; \qquad \alpha^\dagger_{n_1} = \alpha_{n_1}/v. \tag{2.3}$$

Then Eq. (2.2) becomes:

$$\frac{d\rho}{d\xi} + D^\dagger \frac{d^2\rho}{d\xi^2} + \sum_{n_1=0}^{L} \alpha^\dagger_{n_1} \rho^{n_1} = 0. \tag{2.4}$$

Below we shall obtain the exact solution of Eq. (2.4) by application of the modified method of the simplest equation for obtaining exact solutions of nonlinear PDEs. For more details on the modified method of simplest equation see Appendix B.

Proposition 1. *The balance equation for Eq.* (2.4) *for the case when the Riccati equation is used as the simplest equation is* $P(L-1) = 2$ *where* P *is the largest power in the polynomial for* $\rho(\xi)$ *constructed on the basis of the solutions* $\Phi(\xi)$ *of the Riccati equation.*

Proof. We apply the methodology from Appendix B to Eq. (2.4). We constrict a solution as finite series

$$\rho(\xi) = \sum_{i=0}^{P} a_i[\Phi(\xi)]^i, \tag{2.5}$$

where $\Phi(\xi)$ is a solution of the Riccati equation

$$\frac{d\Phi}{d\xi} = a\Phi^2 + b\Phi + c, \tag{2.6}$$

i.e.,

$$\Phi(\xi) = -\frac{b}{2a} - \frac{\theta}{2a}\tanh\left[\frac{\theta(\xi+\xi_0)}{2}\right], \qquad \theta^2 = b^2 - 4ac. \tag{2.7}$$

The substitution of Eq. (2.5) in Eq. (2.4) and the balance of the largest powers of Φ that arise from the different terms of Eq. (2.4) (these powers are $P+2$ from the term $\frac{d^2\rho}{d\xi^2}$ and PL from the term $\alpha_L^\dagger \rho^L$) lead to the balance equation

$$P(L-1) = 2. \quad \square \tag{2.8}$$

Thus we have the possibilities: $P = L = 2$ or $P = 1$; $L = 3$. Below we discuss these possibilities.

2.1.1. Case $P = L = 2$

In this case we shall obtain the exact traveling-wave solution of the equation

$$\frac{\partial \rho}{\partial t} - D\frac{\partial^2 \rho}{\partial x^2} = \alpha_0 + \alpha_1\rho + \alpha_2\rho^2, \tag{2.9}$$

We apply the rescalings from Eq. (2.3) and formulate

Proposition 2. *Suppose that the coefficient* $\alpha_0^\dagger = \frac{625\alpha_1^{\dagger 2} D^{\dagger 2} - 36}{2500\alpha_2^\dagger D^{\dagger 2}} \neq 0$. *Then the traveling wave solution of the kind Eq. (2.5) of Eq. (2.9) obtained on the basis of the modified method of the simplest equation when the equation of Riccati is used as the simplest equation is*

$$\begin{aligned}
\rho(\xi) &= \frac{75D^{\dagger 2}b^2 + 30D^\dagger b - 3 + 25\alpha_1^\dagger D^\dagger}{50\alpha_2^\dagger D^\dagger} + \frac{3[(25D^{\dagger 2}b^2 - 1)(5D^\dagger b + 1)]}{250\alpha_2^\dagger c D^{\dagger 2}} \\
&\quad \times \left\{\frac{b}{2\frac{25D^{\dagger 2}b^2-1}{100cD^{\dagger 2}}} + \frac{\theta}{2\frac{25D^{\dagger 2}b^2-1}{100cD^{\dagger 2}}}\tanh\left[\frac{\theta(\xi+\xi_0)}{2}\right]\right\} - \frac{3(25D^{\dagger 2}b^2-1)}{5000\alpha_2^\dagger c^2 D^{\dagger 3}} \\
&\quad \times \left\{\frac{b}{2\frac{25D^{\dagger 2}b^2-1}{100cD^{\dagger 2}}} + \frac{\theta}{2\frac{25D^{\dagger 2}b^2-1}{100cD^{\dagger 2}}}\tanh\left[\frac{\theta(\xi+\xi_0)}{2}\right]\right\}^2, \\
\theta^2 &= b^2 - \frac{25D^{\dagger 2}b^2 - 1}{25D^{\dagger 2}}.
\end{aligned} \tag{2.10}$$

Suppose that $\alpha_0^\dagger = 0$. *Then the solution is*

$$\begin{aligned}
\rho(\xi) &= -\frac{36b^2 + 60b\alpha_1^\dagger + 25\alpha_1^{\dagger 2}}{100\alpha_1^\dagger\alpha_2^\dagger} + \frac{(36b^2 - 25\alpha_1^{\dagger 2})(6b + 5\alpha_1^\dagger)}{600c\alpha_1^\dagger\alpha_2^\dagger}\left\{\frac{b}{2a} + \frac{\theta}{2a}\tanh\left[\frac{\theta(\xi+\xi_0)}{2}\right]\right\} \\
&\quad - \frac{(36b^2 - 25\alpha_1^{\dagger 2})^2}{14400c\alpha_1^\dagger\alpha_2^\dagger}\left\{\frac{b}{2a} + \frac{\theta}{2a}\tanh\left[\frac{\theta(\xi+\xi_0)}{2}\right]\right\}^2, \\
\theta^2 &= b^2 - \frac{36b^2 - 25\alpha_1^{\dagger 2}}{36}
\end{aligned} \tag{2.11}$$

ξ_0 is a constant of integration.

Proof. As $P = L = 2$ then from Eq. (2.5) the solution will be of the kind

$$\rho(\xi) = a_0 - a_1 \left\{ \frac{b}{2a} + \frac{\theta}{2a} \tanh\left[\frac{\theta(\xi+\xi_0)}{2}\right]\right\} + a_2 \left\{ \frac{b}{2a} + \frac{\theta}{2a} \tanh\left[\frac{\theta(\xi+\xi_0)}{2}\right]\right\}^2. \tag{2.12}$$

The substitution of Eq. (2.12) in Eq. (2.9) leads to a system of relationships for the parameters of the solution (the system is of kind (B.4)):

$$\begin{aligned}
&6D^\dagger a_2 a^2 + \alpha_2^\dagger a_2^2 = 0,\\
&aD^\dagger(a_1 a + 5a_2 b) + a_2 a + \alpha_2^\dagger a_1 a_2 = 0,\\
&D^\dagger[3a_1 ab + 4a_2(2ac + b^2)] + a_1 a + 2a_2 b + \alpha_1^\dagger a_2 + \alpha_2^\dagger(2a_0 a_2 + a_1^2) = 0,\\
&D^\dagger[a_1(2ac + b^2) + 6a_2 bc] + \alpha_1^\dagger a_1 + 2\alpha_2^\dagger a_0 a_1 + a_1 b + 2a_2 c = 0,\\
&\alpha_0^\dagger + \alpha_1^\dagger a_0 + \alpha_2^\dagger a_0^2 + a_1 c + D^\dagger(a_1 bc + 2a_2 c^2) = 0.
\end{aligned} \tag{2.13}$$

Now we have two possibilities: $\alpha_0^\dagger \neq 0$ and $\alpha_0^\dagger = 0$ (which is closer to the classical population dynamics models that usually do not possess terms independent of the population density).

Case $\alpha_0^\dagger \neq 0$

For this case the solution of the system (2.13) is as follows:

$$\begin{aligned}
&\alpha_0^\dagger = \frac{625{\alpha_1^\dagger}^2 {D^\dagger}^2 - 36}{2500\alpha_2^\dagger {D^\dagger}^2}; \qquad a_0 = -\frac{75{D^\dagger}^2 b^2 + 30D^\dagger b - 3 + 25\alpha_1^\dagger D^\dagger}{50\alpha_2^\dagger D^\dagger},\\
&a_1 = -\frac{3[(25{D^\dagger}^2 b^2 - 1)(5D^\dagger b + 1)]}{250\alpha_2^\dagger c D^\dagger}; \qquad a_2 = -\frac{3(25{D^\dagger}^2 b^2 - 1)}{5000\alpha_2^\dagger c^2 {D^\dagger}^3},\\
&a = \frac{25{D^\dagger}^2 b^2 - 1}{100c{D^\dagger}^2}.
\end{aligned} \tag{2.14}$$

and then we obtain the solution (2.10) of Eq. (2.9).

Case $\alpha_0^\dagger = 0$

For this case the solution of the system (2.13) is as follows:

$$\begin{aligned}
&D^\dagger = \frac{6}{25\alpha_1^\dagger}; \qquad a_0 = -\frac{36b^2 + 60b\alpha_1^\dagger + 25{\alpha_1^\dagger}^2}{100\alpha_1^\dagger \alpha_2^\dagger},\\
&a_1 = -\frac{(36b^2 - 25{\alpha_1^\dagger}^2)(6b + 5\alpha_1^\dagger)}{600c\alpha_1^\dagger \alpha_2^\dagger}; \qquad a_2 = -\frac{(36b^2 - 25{\alpha_1^\dagger}^2)^2}{14400c\alpha_1^\dagger \alpha_2^\dagger},\\
&a = \frac{36b^2 - 25{\alpha_1^\dagger}^2}{144c}.
\end{aligned} \tag{2.15}$$

and then we obtain the solution (2.11) of Eq. (2.9). □

The obtained solutions describe kink waves that can be considered as traveling waves of change of the value of the population density of the studied population. Such waves can describe the front of migration of a population. The appropriate values of the boundary conditions ensure that $\rho(\xi)$ is non-negative elsewhere (in order to achieve this the boundary conditions simply have to ensure a positive value for the bottom of the kink). We note that the parameters of the solved Eq. (2.9) are $D^\dagger$ and $\alpha_0^\dagger, \alpha_1^\dagger, \alpha_2^\dagger$. The first relationship from Eq. (2.14) connects these four parameters. Then the solution (2.10) does not hold for any values of parameters of Eq. (2.9) but only for these combinations that satisfy the above mentioned relationship.

Let us discuss in a few words the problem of the boundary conditions of the obtained solutions. Theoretically for the general case of solution (2.10) there are ten parameters and five relationships (2.13) among them. Thus there are five free parameters. A few possibilities for boundary conditions are

$$\begin{aligned}
&\rho(+\infty) = A_1; \qquad \rho(-\infty) = A_2;\\
&\left.\frac{d\rho}{d\xi}\right|_{B_1} = A_3, \quad (B_1 > 0); \qquad \left.\frac{d\rho}{d\xi}\right|_{-B_2} = A_4, \quad (B_2 > 0);\\
&\left.\frac{d^2\rho}{d\xi^2}\right|_{+B_3} = A_5, \quad (B_3 > 0) \qquad \left.\frac{d^2\rho}{d\xi^2}\right|_{-B_4} = A_6, \quad (B_4 > 0);\\
&\ldots
\end{aligned} \tag{2.16}$$

Let us impose the boundary conditions $\rho(+\infty) = A_1;\ \rho(-\infty) = A_2$ on the solution given by Eq. (2.10). The result is that there are two additional relationships that must be satisfied by the parameters of the solution. The relationships are as follows

$$D^\dagger = \frac{6}{25}\frac{1}{\alpha_1^\dagger + 2A_1\alpha_2^\dagger}; \quad \alpha_1^\dagger = -\alpha_2^\dagger(A_1 + A_2). \tag{2.17}$$

The second relationship can be written as follows: $A_2 = -(A_1 + \alpha_1^\dagger/\alpha_2^\dagger)$ which means that if the parameters $\alpha_{1,2}^\dagger$ of the solved PDE are fixed and we also fix A_1 then the boundary condition A_2 cannot be arbitrary. For an example if $A_1 = 0$ then $A_2 = -\alpha_1^\dagger/\alpha_2^\dagger$. A few examples of the obtained nonlinear waves satisfying two boundary conditions are shown in Fig. 1.

2.1.2. Case $P = 1; L = 3$

In this case we shall obtain exact traveling-wave solutions of the equation

$$\frac{\partial \rho}{\partial t} - D\frac{\partial^2 \rho}{\partial x^2} = \alpha_0 + \alpha_1\rho + \alpha_2\rho^2 + \alpha_3\rho^3, \tag{2.18}$$

Proposition 3. *The traveling wave solution of the kind Eq. (2.5) of Eq. (2.18) obtained on the basis of the modified method of the simplest equation when the equation of Riccati is used as the simplest equation is*

$$\rho(\xi) = a_0 - a_1\left\{\frac{b}{2a} + \frac{\theta}{2a}\tanh\left[\frac{\theta(\xi+\xi_0)}{2}\right]\right\} \tag{2.19}$$

for the equation

$$\begin{aligned}\frac{\partial \rho}{\partial t} - D\frac{\partial^2 \rho}{\partial x^2} &= -v\frac{-3aa_0^2D^\dagger a_1 b + 2D^\dagger a^2 a_0^3 - aa_0^2 a_1 - a_1^3 c + a_0 a_1^2 b + 2a_0 D^\dagger a_1^2 ac + a_0 D^\dagger a_1^2 b^2 - D^\dagger a_1^3 bc}{a_1^2} \\ &\quad - v\frac{a_1^2 b + 2D^\dagger a_1^2 ac + D^\dagger a_1^2 b^2 - 6aa_0 D^\dagger a_1 b + 6D^\dagger a^2 a_0^2 - 2aa_0 a_1}{a_1^2}\rho \\ &\quad + v\frac{a(-3D^\dagger a_1 b + 6D^\dagger aa_0 - a_1)}{a_1^2}\rho^2 - 2v\frac{D^\dagger a^2}{a_1^2}\rho^3\end{aligned} \tag{2.20}$$

and

$$\rho(\xi) = \frac{6D^\dagger(-2\alpha_3^\dagger D^\dagger)^{1/2}b - 2\alpha_2^\dagger D^\dagger + (-2\alpha_3^\dagger D^\dagger)^{1/2} + 3(-2\alpha_3^\dagger D^\dagger)^{1/2}D^\dagger\theta\tanh\left[\frac{\theta(\xi+\xi_0)}{2}\right]}{6\alpha_3^\dagger D^\dagger} \tag{2.21}$$

for the equation

$$\begin{aligned}\frac{\partial \rho}{\partial t} - D\frac{\partial^2 \rho}{\partial x^2} &= -\frac{v}{216{\alpha_3^\dagger}^2{D^\dagger}^3}\Big[9bD^\dagger(-2\alpha_3^\dagger D^\dagger)^{3/2} + 27{D^\dagger}^2(-2\alpha_3^\dagger D^\dagger)^{3/2}b^2 + 27{D^\dagger}^3(-2\alpha_3^\dagger D^\dagger)^{3/2}b^3 \\ &\quad + 18(-2\alpha_3^\dagger D^\dagger)^{1/2}D^\dagger\alpha_3 + 24\alpha_2^2{D^\dagger}^2(-2\alpha_3 D^\dagger)^{1/2} + 54(-2\alpha_3 D^\dagger)^{1/2}{D^\dagger}^3\alpha_3 b^2 \\ &\quad - 72(-2\alpha_3 D^\dagger)^{1/2}{D^\dagger}^2\alpha_1^\dagger\alpha_3^\dagger + 16{\alpha_2^\dagger}^3{D^\dagger}^3 + 18(-2\alpha_3^\dagger D^\dagger)^{1/2}b{D^\dagger}^2\alpha_3^\dagger \\ &\quad + 54(-2\alpha_3^\dagger D^\dagger)^{1/2}b^3{D^\dagger}^4\alpha_3^\dagger + (-2\alpha_3^\dagger D^\dagger)^{3/2} - 72\alpha_1^\dagger\alpha_3^\dagger{D^\dagger}^3\alpha_2\Big] + \alpha_1\rho + \alpha_2\rho^2 + \alpha_3\rho^3,\end{aligned} \tag{2.22}$$

Proof. As $P = 1; L = 3$ then from Eq. (2.5) the solution will be of the kind Eq. (2.19) The substitution of Eq. (2.19) in Eq. (2.18) leads to the following system of relationships for the parameters of the solution

$$\begin{aligned}&2D^\dagger a_1 a^2 + \alpha_3^\dagger a_1^3 = 0 \\ &a_1(3D^\dagger ab + \alpha_2^\dagger a_1 + 3\alpha_3^\dagger a_0 a_1 + a) = 0 \\ &a_1 b + D^\dagger a_1(2ac + b^2) + 2\alpha_2^\dagger a_0 a_1 + \alpha_1^\dagger a_1 + 3\alpha_3^\dagger a_0^2 a_1 = 0 \\ &\alpha_2^\dagger a_0^2 + a_1 c + \alpha_1^\dagger a_0 + \alpha_3^\dagger a_0^3 + D^\dagger a_1 bc + \alpha_0^\dagger = 0.\end{aligned} \tag{2.23}$$

We shall consider two possibilities. First we shall solve the system (2.23) with respect to $\alpha_{0,1,2,3}^\dagger$. This means that the coefficients $a_{0,1}$ as well as the coefficients a, b, c of the Riccati equation will remain free and we can impose many boundary conditions on the solution. Thus we shall investigate a small subclass of equations of kind Eq. (2.18) where we have a solution with many free parameters. Then we shall consider the case in which we shall keep as many as possible of the parameters of Eq. (2.18) free. The price for this will be that we shall have to impose a few relationships on the coefficients of the solutions.

Case 1: *Solution of* (2.23) *with respect to* $\alpha^\dagger_{0,1,2,3}$

This solution is as follows

$$\alpha_0^\dagger = -\frac{-3aa_0^2D^\dagger a_1 b + 2D^\dagger a^2 a_0^3 - aa_0^2 a_1 - a_1^3 c + a_0 a_1^2 b + 2a_0 D^\dagger a_1^2 ac + a_0 D^\dagger a_1^2 b^2 - D^\dagger a_1^3 bc}{a_1^2}$$
$$\alpha_1^\dagger = -\frac{a_1^2 b + 2D^\dagger a_1^2 ac + D^\dagger a_1^2 b^2 - 6aa_0 D^\dagger a_1 b + 6D^\dagger a^2 a_0^2 - 2aa_0 a_1}{a_1^2}$$
$$\alpha_2^\dagger = \frac{a(-3D^\dagger a_1 b + 6D^\dagger aa_0 - a_1)}{a_1^2}$$
$$\alpha_3^\dagger = -2\frac{D^\dagger a^2}{a_1^2}. \tag{2.24}$$

Thus we arrive at solution (2.19) of Eq. (2.20).

Case 2: *Solution of* (2.23) *with respect to* a, a_0, a_1 *and* $\alpha_0^\dagger$

The solution is as follows

$$a = \frac{3D^{\dagger 2}\alpha_3 b^2 + 2\alpha_2^{\dagger 2} D^\dagger - 6\alpha_1^\dagger \alpha_3^\dagger D^\dagger + \alpha_3^\dagger}{12cD^{\dagger 2}\alpha_3^\dagger}$$
$$a_0 = \frac{(3D^\dagger b + 1)\sqrt{-2\alpha_3^\dagger D^\dagger} - 2\alpha_2 D^\dagger}{6D^\dagger \alpha_3^\dagger}$$
$$a_1 = \frac{\sqrt{-2\alpha_3^\dagger D^\dagger}(3D^{\dagger 2}\alpha_3^\dagger b^2 + 2\alpha_2^2 D^\dagger - 6\alpha_1^\dagger \alpha_3 D^\dagger + \alpha_3^\dagger)}{12\alpha_3^{\dagger 2} c D^{\dagger 2}}$$
$$\begin{aligned}\alpha_0^\dagger = {} & -\frac{1}{216\alpha_3^{\dagger 2} D^{\dagger 3}}\Big[9bD^\dagger(-2\alpha_3^\dagger D^\dagger)^{3/2} + 27D^{\dagger 2}(-2\alpha_3^\dagger D^\dagger)^{3/2} b^2 + 27D^{\dagger 3}(-2\alpha_3^\dagger D^\dagger)^{3/2} b^3 \\ & + 18(-2\alpha_3^\dagger D^\dagger)^{1/2} D^\dagger \alpha_3 + 24\alpha_2^2 D^{\dagger 2}(-2\alpha_3 D^\dagger)^{1/2} + 54(-2\alpha_3 D^\dagger)^{1/2} D^{\dagger 3}\alpha_3 b^2 \\ & - 72(-2\alpha_3 D^\dagger)^{1/2} D^{\dagger 2}\alpha_1^\dagger \alpha_3^\dagger + 16\alpha_2^{\dagger 3} D^{\dagger 3} + 18(-2\alpha_3^\dagger D^\dagger)^{1/2} bD^{\dagger 2}\alpha_3^\dagger \\ & + 54(-2\alpha_3^\dagger D^\dagger)^{1/2} b^3 D^{\dagger 4}\alpha_3^\dagger + (-2\alpha_3^\dagger D^\dagger)^{3/2} - 72\alpha_1^\dagger \alpha_3^\dagger D^{\dagger 3}\alpha_2\Big].\end{aligned} \tag{2.25}$$

Thus we arrive at solution (2.21) of Eq. (2.22). □

We can now impose boundary conditions of the solution (2.19) of Eq. (2.20). For an example the boundary conditions can be

$$\rho(+\infty) = A_1; \qquad \rho(-\infty) = A_2;$$
$$\left.\frac{d\rho}{d\xi}\right|_{B_1} = A_3, \quad (B_1 > 0); \qquad \left.\frac{d\rho}{d\xi}\right|_{-B_2} = A_4, \quad (B_2 > 0).$$

The boundary conditions will fix additional parameters of the solution. We leave the corresponding algebraic manipulations to the interested reader.

2.2. Coupled waves in a system of three populations

Let us discuss a system of three competing populations modeled by the system of Lotka–Volterra kind (A.7) for the case of constant coefficients of change of population members and constant interaction coefficients, i.e., for the case

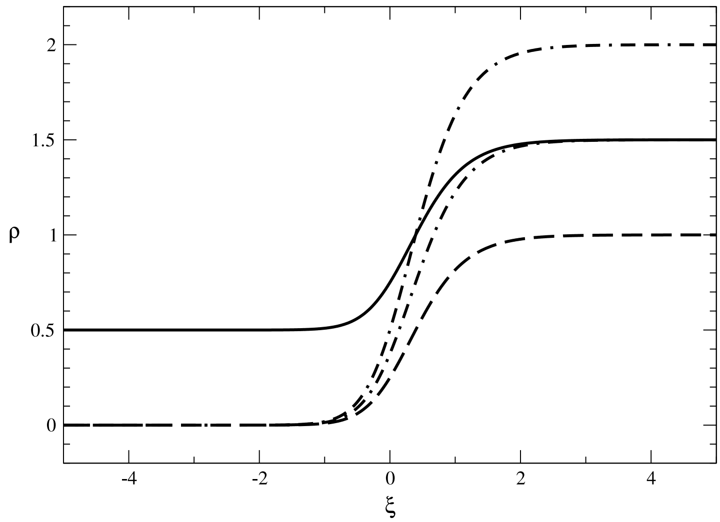

Fig. 1. A few solutions that satisfy boundary conditions $\rho(+\infty) = A_1$; $\rho(-\infty) = A_2$. Solid line: $A_1 = 1.5; A_2 = 0.5$. Dashed line: $A_1 = 1; A_2 = 0$. Dot–dashed line: $A_1 = 1.5; A_2 = 0$. Dot–double dashed line: $A_1 = 2; A_2 = 0$.

$r_{ik} = 0$; $\alpha_{ijk} = 0$. The system of equations becomes

$$\frac{\partial \rho_1}{\partial t} - D_{11}\frac{\partial \rho_1}{\partial x} - D_{12}\frac{\partial \rho_2}{\partial x} - D_{13}\frac{\partial \rho_3}{\partial x} = r_1^0\rho_1 - r_1^0\alpha_{11}^0\rho_1^2 - r_1^0\alpha_{12}^0\rho_1\rho_2 - r_1^0\alpha_{13}^0\rho_1\rho_3$$
$$\frac{\partial \rho_2}{\partial t} - D_{21}\frac{\partial \rho_1}{\partial x} - D_{22}\frac{\partial \rho_2}{\partial x} - D_{23}\frac{\partial \rho_3}{\partial x} = r_2^0\rho_2 - r_2^0\alpha_{21}^0\rho_1\rho_2 - r_2^0\alpha_{22}^0\rho_2^2 - r_2^0\alpha_{23}^0\rho_2\rho_3$$
$$\frac{\partial \rho_3}{\partial t} - D_{31}\frac{\partial \rho_1}{\partial x} - D_{32}\frac{\partial \rho_2}{\partial x} - D_{33}\frac{\partial \rho_3}{\partial x} = r_3^0\rho_3 - r_3^0\alpha_{31}^0\rho_1\rho_3 - r_3^0\alpha_{32}^0\rho_2\rho_3 - r_3^0\alpha_{33}^0\rho_3^2. \tag{2.26}$$

For this system we shall demonstrate the existence of a simple coupled kink wave solution (more complicated solutions are possible too). We note that in real situations $D_{ii} = 0, i = 1, 2, 3$ but above we allow the possibility that D_{ii} is also non-negative in order to obtain a more general solution of Eqs. (2.26).

Proposition 4. *The system (2.26) possesses a coupled kind of wave solution of the type*

$$\rho_1(\xi) = -\frac{a_1c_0}{a_1+b_1} + a_1\left\{\frac{a_1+b_1+4a_1c_0}{4a_1(a_1+b_1)} + \frac{\theta(a_1+b_1+4a_1c_0)}{4a_1b(a_1+b_1)}\tanh\left[\frac{\theta(\xi+\xi_0)}{2}\right]\right\},$$
$$\rho_2(\xi) = -\frac{b_1c_0}{a_1+b_1} + b_1\left\{\frac{a_1+b_1+4a_1c_0}{4a_1(a_1+b_1)} + \frac{\theta(a_1+b_1+4a_1c_0)}{4a_1b(a_1+b_1)}\tanh\left[\frac{\theta(\xi+\xi_0)}{2}\right]\right\},$$
$$\rho_3(\xi) = -(a_1+b_1) + c_1\left\{\frac{a_1+b_1+4a_1c_0}{4a_1(a_1+b_1)} + \frac{\theta(a_1+b_1+4a_1c_0)}{4a_1b(a_1+b_1)}\tanh\left[\frac{\theta(\xi+\xi_0)}{2}\right]\right\}, \tag{2.27}$$

where $\xi - x + \frac{4a_1c_0+(a_1+b_1)(1-b+bD_{11})}{b(a_1+b_1)}t$ *and* a_1, b, b_1, c_0, D_{11} *are free parameters.*

We remember that above $\theta^2 = b^2 - 4ac$ where a, b, c are the parameters of the Riccati equation.

Proof. We apply the modified method of the simplest equation to the system (2.26). First of all we introduce the traveling wave coordinate $\xi = x - vt$ where v is the wave velocity. Then we search for the solution in the form

$$\rho_1(\xi) = \sum_{i=0}^{P} a_i\Phi(\xi)^i; \qquad \rho_2(\xi) = \sum_{j=0}^{Q} b_j\Phi(\xi)^j; \qquad \rho_3(\xi) = \sum_{k=0}^{R} c_k\Phi(\xi)^k \tag{2.28}$$

where $\frac{d\Phi}{d\xi} = a\Phi^2 + b\Phi + c$. The simplest possible balance equation is $P = Q = R = 1$. The substitution of all the above in the system (2.26) leads to the following system of nine nonlinear algebraic equations

$$
\begin{aligned}
&(1)\ -D_{13}c_1a-(v+D_{11})a_1a-r_{10}\alpha_{120}a_1b_1+r_{10}\alpha_{110}a_1^2-D_{12}b_1a-r_{10}\alpha_{130}a_1c_1=0\\
&(2)\ -r_{10}\alpha_{120}a_0b_1-r_{10}\alpha_{120}a_1b_0+2r_{10}\alpha_{110}a_0a_1-r_{10}\alpha_{130}a_0c_1-r_{10}\alpha_{130}a_1c_0-r_{10}a_1\\
&\quad -D_{13}c_1b-D_{12}b_1b-(v+D_{11})a_1b=0\\
&(3)\ -(v+D_{11})a_1c-r_{10}\alpha_{120}a_0b_0-D_{13}c_1c-D_{12}b_1c-r_{10}\alpha_{130}a_0c_0-r_{10}a_0+r_{10}\alpha_{110}a_0^2=0\\
&(4)\ -D_{23}c_1a-D_{21}a_1a-r_{20}\alpha_{220}b_1^2+r_{20}\alpha_{210}a_1b_1-(v+D_{22})b_1a-r_{20}\alpha_{230}b_1c_1=0\\
&(5)\ -2r_{20}\alpha_{220}b_0b_1+r_{20}\alpha_{210}a_0b_1+r_{20}\alpha_{210}a_1b_0-r_{20}\alpha_{230}b_0c_1-r_{20}\alpha_{230}b_1c_0-r_{20}b_1\\
&\quad -D_{23}c_1b-(v+D_{22})b_1b-D_{21}a_1b=0\\
&(6)\ -D_{21}a_1c-r_{20}\alpha_{220}b_0^2-D_{23}c_1c-(v+D_{22})b_1c-r_{20}\alpha_{230}b_0c_0-r_{20}b_0+r_{20}\alpha_{210}a_0b_0=0\\
&(7)\ -(v+D_{33})c_1a-D_{31}a_1a-r_{30}\alpha_{320}b_1c_1+r_{30}\alpha_{310}a_1c_1-D_{32}b_1a-r_{30}\alpha_{330}c_1^2=0\\
&(8)\ -r_{30}\alpha_{320}b_0c_1-r_{30}\alpha_{320}b_1c_0+r_{30}\alpha_{310}a_0c_1+r_{30}\alpha_{310}a_1c_0-2r_{30}\alpha_{330}c_0c_1-r_{30}c_1\\
&\quad -(v+D_{33})c_1b-D_{32}b_1b-D_{31}a_1b=0\\
&(9)\ -D_{31}a_1c-r_{30}\alpha_{320}b_0c_0-(v+D_{33})c_1c-D_{32}b_1c-r_{30}\alpha_{330}c_0^2-r_{30}c_0+r_{30}\alpha_{310}a_0c_0=0.
\end{aligned}
\tag{2.29}
$$

The general solution of this system is very long. In order to obtain the solution from the text of Proposition 4 we fix some of the parameters in the above system as follows

$$
\begin{aligned}
&r_{10}=1;\quad r_{20}=1;\quad r_{30}=1;\quad \alpha_{110}=1;\\
&\alpha_{220}=1;\quad \alpha_{330}=1;\quad \alpha_{120}=1;\quad \alpha_{130}=1;\quad \alpha_{210}=1;\\
&\alpha_{230}=1;\quad \alpha_{310}=1;\quad \alpha_{320}=1;\\
&D_{21}=D_{12};\quad D_{31}=D_{13};\quad D_{32}=D_{23};\quad D_{22}=D_{11};\quad D_{33}=D_{11};\\
&D_{12}=1;\quad D_{13}=1;\quad D_{23}=1.
\end{aligned}
\tag{2.30}
$$

In this simple case the solution of the system (2.29) is (for details of solution of the system of equations by means of a Maple program see Appendix C)

$$
\begin{aligned}
c_1&=-(a_1+b_1)\\
v&=-\frac{4a_1c_0+bD_{11}a_1-a_1b+a_1-b_1b+bD_{11}b_1+b_1}{b(a_1+b_1)}\\
c&=-\frac{c_0(2a_1c_0+a_1+b_1)b}{(a_1+b_1)(a_1+b_1+4a_1c_0)}\\
a&=-2\frac{a_1b(a_1+b_1)}{(a_1+b_1+4a_1c_0)}\\
a_0&=-\frac{a_1c_0}{a_1+b_1}\\
b_0&=-\frac{b_1c_0}{a_1+b_1}.
\end{aligned}
\tag{2.31}
$$

Substituting the coefficients in the functions ρ_1, ρ_2, ρ_3 we obtain the coupled kink wave solution from the formulation of the Proposition 4. □

3. Statistical distributions and exit time

Eqs. (A.11) and (A.12) are typical equations for description of the evolution of dynamical systems. The general case of such equations is

$$
\frac{dx_i}{dt}=X_i(x_1,x_2,\ldots,x_N);\quad i=1,2,\ldots,N \tag{3.1}
$$

where $X_i(x_1, x_2, \ldots, x_N)$ is some (in the general case nonlinear) function. For such a kind of systems there exists a theory that allows us to characterize some system properties in the case when the system is under the action of random perturbations. Pontryagin, Andronov and Vitt [19] developed such a theory for random impulses that occur after every interval of time τ and each impulse causes the phase point of the dynamical system described by Eqs. (3.1) to jump through a distance a along a random direction. Let us first consider the case of a single population and one spatial dimension. For the case when a tends to zero together with τ in such a way that the ratio a^3/τ tends to a finite limit b it is possible to obtain an equation for the probability density function $p(x,t)$ as follows (for more discussion see Appendix D):

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x}[X(x)p] = \frac{b}{2}\frac{\partial^2 p}{\partial x^2}. \tag{3.2}$$

For the general case of N populations the equation for the probability density function becomes

$$\frac{\partial p}{\partial t} + \sum_{i=1}^{N} \frac{\partial}{\partial x_i}[X_i(x_1, x_2, \dots, x_N)p] = \frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} b_{ij}\frac{\partial^2 p}{\partial x_i \partial x_j}, \tag{3.3}$$

where b_{ij} are again coefficients that characterize the random impulses.

Another kind of problem that can be solved by this approach is to calculate the mathematical expectation of the exit time. Let us again first discuss the case of one population and one spatial dimension. We have a phase point that is inside the interval $[\epsilon_1, \epsilon_2]$ $(\epsilon_1 < \epsilon_2)$ and the system is under the influence of the same random perturbations as described above. The exit time is the time for which the phase point that was inside the above interval at $t = 0$ will leave this interval through ϵ_1 or through ϵ_2. If we denote as $F(x)$ the mathematical expectation for the exit time then $F(x)$ is a solution of the equation [19,20]

$$\frac{b}{2}\frac{d^2 F}{dx^2} + X(x)\frac{dF}{dx} + 1 = 0, \tag{3.4}$$

with boundary conditions $F(\epsilon_1) = F(\epsilon_2) = 0$. For the case of many populations the zero boundary conditions are on the entire border of the multidimensional phase space area that has to be exited and the equation for the probability density function of the exit time is

$$\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N} b_{ij}\frac{\partial^2 F}{\partial x_i \partial x_j} + \sum_{i=1}^{N} X(x_1, x_2, \dots, x_N)\frac{\partial F}{\partial x_i} + 1 = 0. \tag{3.5}$$

Let us now apply this theory to Eq. (A.12). We shall be interested in the stationary distributions $p(\overline{\rho})$ connected to Eq. (A.12), i.e., we shall be interested in the case when after a long time the probability density function p becomes stationary and depends only on the spatial coordinate $\overline{\rho}$. This stationary case is important because of Observation 2 from Appendix D which states that each time dependent solution $p(x,t)$ of the Fokker–Planck equation (D.7) converges at $t \to \infty$ to the stationary distribution $p_0(x)$ from Eq. (D.12). In our case $x = \overline{\rho}$ and $X(x) = \sum_{n_1=0}^{L} \alpha_{n_1}\overline{\rho}^{n_1}$. We shall discuss two cases: (i) the case of a single population where $\overline{\rho} \in [0, \infty)$; and (ii) another case not connected directly to the population dynamics where $\overline{\rho} \in (-\infty, \infty)$.

Case 1: $\overline{\rho} \in [0, \infty)$

This case is connected directly to the population dynamics as the population density cannot be negative. We formulate

Proposition 5. *Let us discuss a system described by the equation*

$$\frac{d\overline{\rho}}{dt} = \sum_{n_1=0}^{L} \alpha_{n_1}\overline{\rho}^{n1} \tag{3.6}$$

which is under the action of random impulses that occur after every interval of time τ and each impulse causes the phase point of the dynamical system described by Eqs. (3.12) to jump through a distance a along a random direction. Let a tend to zero together with τ in such a way that the ratio a^3/τ tends to a finite limit b. Let in addition the following requirements be fulfilled

$$p(0) = 0; \quad \rho \in [0, \infty). \tag{3.7}$$

Then the stationary p.d.f. $p(\overline{\rho})$ is

$$p(\overline{\rho}) = C\exp\left[\frac{2}{b}\sum_{n_1=0}^{L}\alpha_{n_1}\frac{\overline{\rho}^{n_1+1}}{n_1+1}\right]\left\{1 - \int d\overline{\rho}\,\exp\left[-\frac{2}{b}\sum_{n_1=0}^{L}\alpha_{n_1}\frac{\overline{\rho}^{n_1+1}}{n_1+1}\right]\right\}, \tag{3.8}$$

where the constant of integration C is determined by the normalization condition

$$\int_{-\infty}^{\infty} d\overline{\rho}\, p(\overline{\rho}) = 1. \tag{3.9}$$

Proof. The equation for the stationary distribution $p(\overline{\rho})$ is a particular case of Eq. (3.2). One integration of the equation for the stationary distribution leads to

$$\frac{b}{2}\frac{dp}{d\overline{\rho}} - \left[p(\overline{\rho})\sum_{n_1=0}^{L}\alpha_{n_1}\overline{\rho}^{n1}\right] + C_1 = 0 \tag{3.10}$$

Eq. (3.10) can be easily integrated and the result is

$$p(\overline{\rho}) = \exp\left[\frac{2}{b}\sum_{n_1=0}^{L}\alpha_{n_1}\frac{\overline{\rho}^{n_1+1}}{n_1+1}\right]\left\{C - \frac{2C_1}{b}\int d\overline{\rho}\ \exp\left[-\frac{2}{b}\sum_{n_1=0}^{L}\alpha_{n_1}\frac{\overline{\rho}^{n_1+1}}{n_1+1}\right]\right\}, \tag{3.11}$$

where C is a constant of integration. The condition $p(0) = 0$ leads to $C_1 = bC/2$ and the distribution described by Eq. (3.11) is reduced to the distribution from Eq. (3.8). □

Case 2: $\overline{\rho} \in (-\infty, \infty)$

Proposition 6. *Let us discuss a system described by the equation*

$$\frac{d\overline{\rho}}{dt} = \sum_{n_1=0}^{L}\alpha_{n_1}\overline{\rho}^{n1} \tag{3.12}$$

which is under the action of random impulses that occur after every interval of time τ and each impulse causes the phase point of the dynamical system described by Eqs. (3.12) to jump through a distance a along a random direction. Let a tend to zero together with τ in such a way that the ratio a^3/τ tends to a finite limit b.

- ⋆ *Subcase 1:*

  Let in addition the following requirements be fulfilled

$$p(0) = A; \quad \overline{\rho} \in (-\infty, \infty). \tag{3.13}$$

  Then the stationary p.d.f. $p(\overline{\rho})$ is

$$p(\overline{\rho}) = \exp\left[\frac{2}{b}\sum_{n_1=0}^{L}\alpha_{n_1}\frac{\overline{\rho}^{n_1+1}}{n_1+1}\right]\left\{C - (C-A)\int d\overline{\rho}\ \exp\left[-\frac{2}{b}\sum_{n_1=0}^{L}\alpha_{n_1}\frac{\overline{\rho}^{n_1+1}}{n_1+1}\right]\right\}, \tag{3.14}$$

  where the constant of integration C is determined by the normalization condition

$$\int_{-\infty}^{\infty} d\overline{\rho}\ p(\overline{\rho}) = 1. \tag{3.15}$$

- ⋆⋆ *Subcase 2:*

  Let in addition the following requirements be fulfilled

$$\frac{1}{p(0)}\frac{dp}{d\overline{\rho}}\bigg|_{\overline{\rho}=0} = \frac{2\alpha_0}{b}; \quad \overline{\rho} \in (-\infty, \infty). \tag{3.16}$$

  Then the stationary p.d.f. $p(\overline{\rho})$ is

$$p(\overline{\rho}) = C\exp\left[\frac{2}{b}\sum_{n_1=0}^{L}\alpha_{n_1}\frac{\overline{\rho}^{n_1+1}}{n_1+1}\right], \tag{3.17}$$

  where the constant of integration C is determined by the normalization condition

$$\int_{-\infty}^{\infty} d\overline{\rho}\ p(\overline{\rho}) = 1. \tag{3.18}$$

Proof. *Subcase* 1

The equation of the stationary distribution $p(\overline{\rho})$ is a particular case of Eq. (3.2). One integration of the equation for the stationary distribution leads to Eq. (3.10). Eq. (3.10) can be easily integrated and the result is Eq. (3.11) where C is a constant of integration. The condition $p(0) = A$ leads to $C_1 = b(C - A)/2$ and the distribution described by Eq. (3.11) is reduced to the distribution from Eq. (3.14).

Subcase 2

The integration of Eq. (3.2) leads to Eq. (3.10) where C_1 is a constant of integration. This constant is equal to zero because of the condition (3.16). With $C_1 = 0$ we can continue the integration of Eq. (3.10) and the result is (3.17) where the constant of integration C is determined by the normalization condition (3.18). □

We note that Eq. (3.16) in combination with Eq. (3.17) mean that $\sum_{n_1=0}^{L}\alpha_{n_1} = 0$. In addition $f(\overline{\rho})$ must tend to zero when $\overline{\rho} \to \pm\infty$. The dominant term at large values of $\overline{\rho}$ is $\alpha_L\overline{\rho}^L$. Then α_L must be negative (to ensure $f \to 0$ at large positive values of $\overline{\rho}$) and L must be odd (to ensure $f \to 0$ at large negative values of $\overline{\rho}$).

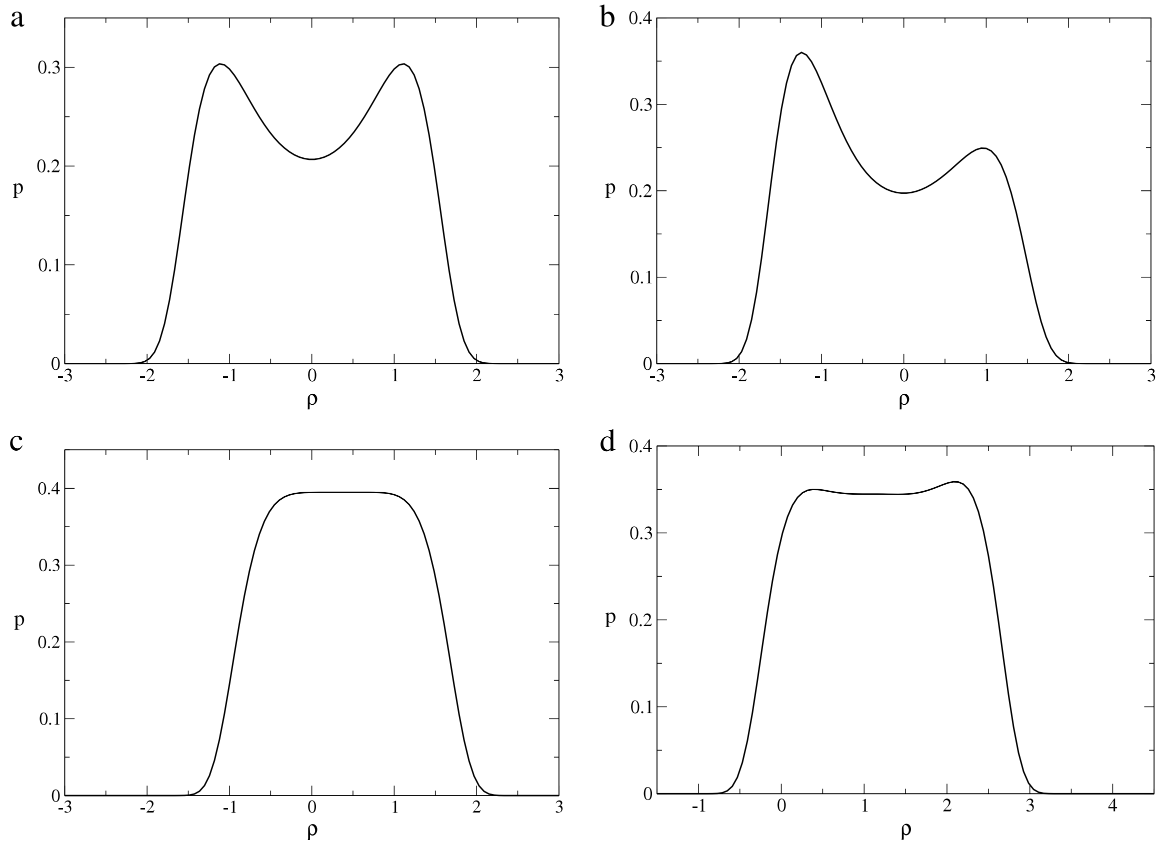

Fig. 2. A few profiles of $p(\overline{\rho})$ from Eq. (3.17) (in the figures $\overline{\rho}$ is denoted as ρ). (a) $b = 2$; $\alpha_0 = 0$; $\alpha_1 = 1$; $\alpha_2 = -0.4$; $\alpha_3 = 0.2$; $\alpha_4 = 0$; $\alpha_5 = -0.5$. (b) $\alpha_0 = 0$; $\alpha_1 = 1$; $\alpha_2 = 0$; $\alpha_3 = 0.2$; $\alpha_4 = 0$; $\alpha_5 = -0.5$; $b = 2$. Here there are three fixed points. The two maxima of the p.d.f. distribution are centered around the two stable fixed points and the minimum is centered on the unstable fixed point $\overline{\rho} = 0$. (c) $\alpha_0 = 0.003564$; $\alpha_1 = -0.006084$; $\alpha_2 = 0.3975$; $\alpha_3 = -1.234$; $\alpha_4 = 1.81$; $\alpha_5 = -1$; $b = 2$. (d) $\alpha_0 = 1.2936$; $\alpha_1 = -7.2436$; $\alpha_2 = 14.58$; $\alpha_3 = -13.63$; $\alpha_4 = 6$; $\alpha_5 = -1$; $b = 2$. The probability distribution has three maxima (at $\overline{\rho} = 0.4, 1.1, 2.1$) and two minima.

A few examples of statistical distributions $f(\overline{\rho})$ are shown in Fig. 2. Let us note here the tunnel phenomenon which arises because of the presence of fluctuations. The existence of the distributions $p(\rho, t)$ and $p_0(\rho)$ means that in the course of time each value of the density ρ can be reached. Then if for example at the initial moment the system trajectory in the phase space is close to a fixed point of the ODE without added fluctuations then in the course of time the phase point can leave this area and travel to a phase space area that is close to another fixed point. This tunnel phenomenon is closely connected to the role played by fluctuations in the case when a bifurcation happens in the studied system.

Let us now calculate the exit time expectation on the basis of Eq. (3.4).

Proposition 7. *Let us discuss the system described by Eq. (3.12). The distribution $F_q(\overline{\rho})$ from the initial position $\overline{\rho}$ to the position $q < \overline{\rho}$ is*

$$F_q(\overline{\rho}) = \int_q^{\overline{\rho}} d\xi \; \exp\left(-\frac{2}{b}\sum_{n_1=0}^{L}\alpha_{n_1}\frac{\xi^{n_1+1}}{n_1+1}\right)\left[\frac{2}{b}\int_{\xi}^{\infty} d\eta \; \exp\left(\frac{2}{b}\sum_{n_1=0}^{L}\alpha_{n_1}\frac{\eta^{n_1+1}}{n_1+1}\right)\right] \tag{3.19}$$

when

($\star$) $F(\overline{\rho} = q) = 0$,

($\star\star$) *$F(\overline{\rho}, q)$ increases in the slowest possible manner as $\overline{\rho} \to \infty$.*

Proof. We calculate the distribution for exit time from the initial position $\overline{\rho}$ to a position $q < \overline{\rho}$. In the discussed case again $X(x) = \sum_{n_1=0}^{L}\alpha_{n_1}\overline{\rho}^{n_1}$. One integration of Eq. (3.4) leads to the equation

$$\frac{dF}{d\overline{\rho}} = \exp(-\psi(\overline{\rho}))\left(C_1 + \int_{\overline{\rho}}^{\infty} d\xi \; \frac{2}{b}\exp(\psi(\xi))\right); \quad \psi(\overline{\rho}) = \frac{2}{b}\int_{\overline{\rho}}^{\infty} d\xi X(\xi). \tag{3.20}$$

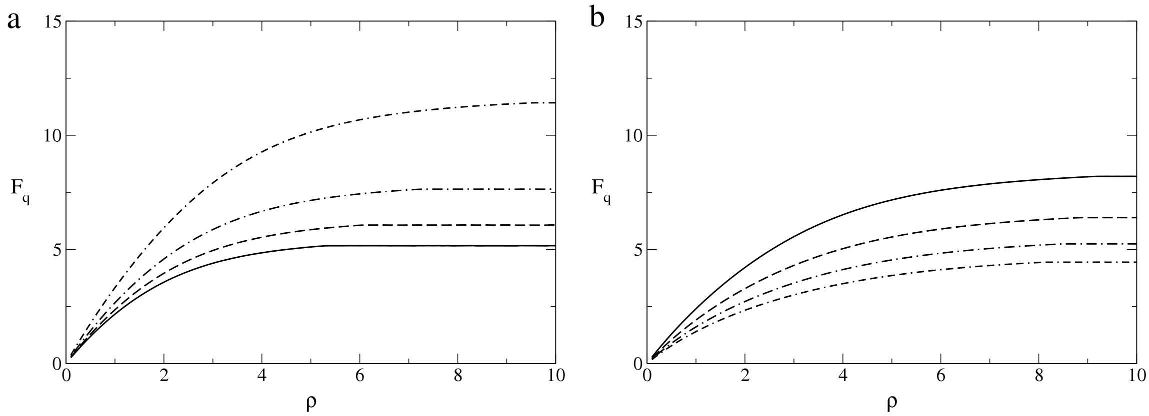

Fig. 3. Exit time expectations for $q = 0$ (which means extinction of the population) calculated on the basis of Eq. (3.19). ρ on the horizontal axis is equal to $\overline{\rho}$ from Eq. (3.19). For all curves $b = 2$. (a) Influence of the value of α_3 on the exit time. $\alpha_0 = 0.01$; $\alpha_1 = 0.2$; $\alpha_2 = 0.1$. Solid line: $\alpha_3 = -0.05$; dashed line: $\alpha_3 = -0.04$; dot–dashed line: $\alpha_3 = -0.03$; dot–double dashed line: $\alpha_3 = -0.02$. (b) Influence of the value of α_1 on the exit time. $\alpha_0 = 0.01$; $\alpha_2 = 0.1$; $\alpha - 3 = -0.02$. Solid line: $\alpha_1 = -0.3$; dashed line: $\alpha_1 = -0.4$; dot–dashed line: $\alpha_1 = -0.5$; dot–double dashed line: $\alpha_1 = -0.6$.

The relationship (**) requires $C_1 = 0$ (as the corresponding term in Eq. (3.20) vanishes and the growth of $\frac{dF}{d\bar{\rho}}$ is as slow as possible), and the integration of Eq. (3.20) leads to the result (3.19). □

Fig. 3 shows the dependence of the exit time expectation on the population density and coefficients of the model equation for the case $L = 3$. The negative values of $\alpha_{1,3}$ make extinction expected sooner whereas the positive values of the other two parameters can delay the extinction. The theory can be easily applied for the case of a system of many interacting populations but even in the simplest one-dimensional case the integral from Eq. (3.19) must be calculated numerically.

4. Conclusion

In this paper we have discussed two aspects of population dynamics. First we have presented a model of the space–time dynamics of the interacting population system in two spatial dimensions. For the simplest case of one spatial dimension and for one population we have obtained an exact traveling-wave solution of the model nonlinear PDE by means of the recently developed modified method of the simplest equation for obtaining exact and approximate solutions of nonlinear PDEs. The obtained exact solution describes the propagation of changes of the population density in the space. The generalization of this theory to the case of many populations is straightforward and describes the spreading of coupled waves of changes of densities of the studied populations. The case of three populations is discussed in the paper. The second discussed aspect of the population dynamics was connected to the influence of the random fluctuations on the population densities. The presence of fluctuations leads to description in terms of probability density functions for the population densities. The discussed general theory is illustrated again for the simplest possible case of one population in two aspects: calculation of probability density functions and calculation of the expected extinction time. As expected from the theory of diffusion Markov processes the minima and maxima of the obtained probability density functions are exactly at the fixed points of the corresponding non-perturbed model system of differential equations. The expected extinction time strongly depends on the coefficients of the model equations. Finally a few results are obtained that are not connected to the theory of interacting populations but may be interesting for other cases modeled by nonlinear PDEs with polynomial nonlinearities.

Acknowledgment

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Appendix A. The model equations

A.1. Spatially distributed populations

Let us consider a two-dimensional area S where N competing populations are present. The density of each population is $\rho_i(x, y, t) = \frac{\Delta N_i}{\Delta S}$, where ΔN_i is the number of individuals of the ith population that are present in the small area ΔS at the moment t. Now let a movement of population members through the borders of the area ΔS be possible and let $\vec{j}_i(x, y, t)$ be the current of this movement. Then $(\vec{j}_i \cdot \vec{n})\delta l$ is the net number of members from the ith population, crossing a small border line δl with normal vector n. Let the density changes be summarized by the function $C_i(\rho_1, \rho_2, \dots, \rho_N, x, y, t)$. Then the change of the density of members of the ith population in the studied area is described by the equation

$$\frac{\partial \rho_i}{\partial t} + \mathrm{div}\vec{j}_i = C_i. \tag{A.1}$$

Below we shall discuss the case where $\vec{j}_i$ has the form of linear multicomponent diffusion. In this case

$$\vec{j}_i = -\sum_{k=1}^{N} D_{ik}(\rho_i, \rho_k, x, y, t)\nabla \rho_k, \tag{A.2}$$

where D_{ik} is the diffusion coefficient. Eq. (A.2) reflects the possibility that the motion of the population members is caused not only by gradients of the density of the own population but also it could be caused by gradients of the densities of the other populations.

In this paper we shall consider the case where the C_i depend only on the population densities ρ_k, $k = 1, \dots, N$. We shall not specify the kind of the function C_i as we shall consider the general case of relatively small population densities that allow us to write C_i as Taylor series expansion around the zero values of all population densities as follows

$$C_i(\rho_1, \rho_2, \dots, \rho_N) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\dots\sum_{n_N=0}^{\infty} \alpha^{(i)}_{n_1,n_2,\dots,n_N} \rho_1^{n_1}\rho_2^{n_2}\dots\rho_N^{n_N}, \tag{A.3}$$

where the constant coefficients $\alpha^{(i)}_{n_1,n_2,\dots,n_N}$ are as follows

$$\alpha^{(i)}_{n_1,n_2,\dots,n_N} = \frac{1}{n_1!n_2!\dots n_N!}\left.\frac{\partial C_i^{n_1+n_2+\dots+n_N}}{\partial\rho_1^{n_1}\partial\rho_2^{n_2}\dots\partial\rho_N^{n_N}}\right|_{\rho_1=\rho_2=\dots=\rho_N=0}. \tag{A.4}$$

For an example let us consider the one-dimensional case and in addition let the diffusion coefficients D_{ik} be constants. Then the substitution of Eqs. (A.2) and (A.3) in (A.1) leads to the following system of nonlinear PDEs for the studied N interacting populations:

$$\frac{\partial \rho_i}{\partial t} - \sum_{k=1}^{N} D_{ik}\frac{\partial^2\rho_k}{\partial x^2} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\dots\sum_{n_N=0}^{\infty} \alpha^{(i)}_{n_1,n_2,\dots,n_N} \rho_1^{n_1}\rho_2^{n_2}\dots\rho_N^{n_N}. \tag{A.5}$$

For the case of one population the system (A.5) is reduced to the equation

$$\frac{\partial \rho}{\partial t} - D\frac{\partial^2\rho}{\partial x^2} = \sum_{n_1=0}^{\infty}\alpha_{n_1}\rho^{n_1}. \tag{A.6}$$

Let us give two examples of systems of the kind (A.5). The first example is connected to the classical Lotka–Volterra case extended in [14]. In this case after assumption of dependence of the growth rates and competition coefficients on the population density one arrives at the system of equations

$$\frac{\partial \rho_i}{\partial t} - \sum_{j=1}^{n} D_{ij}\Delta\rho_j = r_i^0\rho_i\left[1 - \sum_{j=1}^{n}(\alpha_{ij}^0 - r_{ij}\rho_j) - \sum_{j,k=1}^{n}\alpha_{ij}^0(\alpha_{ijk}+\rho_k)\rho_j\rho_k - \sum_{j,k,l=1}^{n}\alpha_{ij}^0 r_{ik}\alpha_{ijl}r_{ik}\alpha_{ijl}\rho_j\rho_k\rho_l\right] \tag{A.7}$$

where $i = 1, 2, \dots, n$ is a number that indexes the n competing populations; D_{ij} is the diffusion coefficient; $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$; r_i^0 and α_{ij}^0 are the parts of the corresponding growth rates and competition coefficients that do not depend on the population densities; r_{ik} and α_{ijk} are parameters that regulate the intensity of the dependence of the population growth rates and competition coefficients on the population densities ρ_i. It can be easily checked that the system described by Eq. (A.7) is a particular case of the system of equations given by Eqs. (A.1)–(A.4).

A system of a kind similar to the system from Eq. (A.7) arises in the social dynamics in the spatial model of ideological struggle developed in [21]. For this case the model system of equations is

$$\frac{\partial \rho_i}{\partial t} - \sum_{j=1}^{n} D_{ij}\Delta\rho_j = r_i\rho_i + \sum_{j=1}^{n} f_{ij}\rho_j + \sum_{j=1}^{n}\alpha_{ij}\rho_i\rho_j + \sum_{j,k=1}^{n} b_{ijk}\rho_i\rho_j\rho_k + \cdots \tag{A.8}$$

where $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$; ρ_i, $i = 1, 2, \dots, n$ are the spatial densities of the populations of the followers of the corresponding ideology; r_i are the rates of change of corresponding populations of adepts by births and deaths; f_{ij} is the coefficient of non-contact conversion (the ideology of a person can be changed without contact between humans but by mass media influence for an example); a_{ij} is the coefficient of binary contact conversion that describes the change of ideology by contacts between followers of different ideologies.

A.2. Spatially averaged equations

Below we shall apply spatial averaging to the system of Eqs. (A.5) similar to the averaging used in the optimum theory of turbulence [22,23]. In the general two-dimensional case let a quantity $q(x, y, t)$ be defined in a large two-dimensional plane area S with acreage $|S|$. Then by definition the spatial average of q is

$$\overline{q}(t) = \frac{1}{|S|} \int\int_S dx\, dy\, q(x, y, t). \tag{A.9}$$

$q(x, y, t)$ can be separated into a spatially averaged part $\overline{q}$ and the remainder $Q(x, y, t)$:

$$q(x, y, t) = \overline{q}(t) + Q(x, y, t). \tag{A.10}$$

Let the plane average of any product of the remainder quantities vanish: $\overline{Q_i} = \overline{Q_iQ_j} = \overline{Q_iQ_jQ_k} = \cdots = 0$. In addition we shall assume that $\int\int_S dx\, dy \nabla^2 Q$ has finite and small value such that $\overline{\nabla^2 Q} = \frac{1}{|S|}\int\int_S dx\, dy \nabla^2 Q \to 0$. The application of the averaging to Eq. (A.1) in the presence of the assumptions given by Eqs. (A.2)–(A.4) (note that in this case we have two spatial dimensions) leads to the system of ODEs as follows ($i = 1, 2, \dots, N$):

$$\frac{d\overline{\rho}_i}{dt} = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_N=0}^{\infty} \alpha^{(i)}_{n_1,n_2,\dots,n_N} \overline{\rho}_1^{n_1}\overline{\rho}_2^{n_2}\dots\overline{\rho}_N^{n_N}. \tag{A.11}$$

We note that Eq. (A.11) follows directly from Eq. (A.5) in the spatially homogeneous case. The above discussion shows that Eq. (A.11) can also arise in the spatially inhomogeneous case. For the case of one population Eq. (A.11) becomes

$$\frac{d\overline{\rho}}{dt} = \sum_{n_1=0}^{\infty} \alpha_{n_1}\overline{\rho}^{n_1}. \tag{A.12}$$

We note that equations of the kind Eqs. (A.11) and (A.12) are often used as model equations in population dynamics not only for small values of population densities but also for large values of these densities, i.e., for large $\overline{\rho}_i$. One example is connected to Holling functional response functions in predator–prey systems [24]. For the case of one predator and one prey the functional response can be for an example type II Holling functional response: $f(\overline{\rho}) = \frac{a\overline{\rho}}{1+ah\overline{\rho}}$ where $\overline{\rho}$ is the prey density and a and h are parameters. The functional response can be also a type III Holling functional response: $f(\overline{\rho}) = \frac{\overline{\rho}^2}{h+\overline{\rho}^2}$. The functional response function for the case of many prey species can be more complicated.

For small values of population densities even the models with complicated nonlinear functional responses can be reduced to models with polynomial nonlinearities. An example is the one predator–two prey model [25]

$$\begin{aligned}
\frac{d\overline{\rho}_1}{dt} &= \overline{\rho}_1 g_1(\overline{\rho}_1, \overline{\rho}_2) - \overline{\rho}_3 f_1(\overline{\rho}_1, \overline{\rho}_2)\\
\frac{d\overline{\rho}_2}{dt} &= \overline{\rho}_2 g_2(\overline{\rho}_1, \overline{\rho}_2) - \overline{\rho}_3 f_2(\overline{\rho}_1, \overline{\rho}_2)\\
\frac{d\overline{\rho}_3}{dt} &= \overline{\rho}_3[c_1 f_1(\overline{\rho}_1, \overline{\rho}_2) + c_2 f_2(\overline{\rho}_1, \overline{\rho}_2)] - m\overline{\rho}_3
\end{aligned} \tag{A.13}$$

where f_i are the functional responses; m is the constant mortality rate of the predator; c_i are the conversion factors of captured prey species into predators and g_i are the growth functions of the corresponding prey type. $\overline{\rho}_{1,2}$ are the spatial densities of the two types of prey and $\overline{\rho}_3$ is the spatial density of the predator species. For small population densities we can apply Taylor series expansion for the functions $f_{1,2}$ and $g_{1,2}$ and thus obtain a system of equations from the class of equations studied in this paper

$$\begin{aligned}
\frac{d\overline{\rho}_1}{dt} &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{\overline{\rho}_1^{n_1}\overline{\rho}_2^{n_2}}{n_1!n_2!}(\overline{\rho}_1\alpha_{3,n_1,n_2} - \overline{\rho}_3\alpha_{1,n_1,n_2})\\
\frac{d\overline{\rho}_2}{dt} &= \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{\overline{\rho}_1^{n_1}\overline{\rho}_2^{n_2}}{n_1!n_2!}(\overline{\rho}_2\alpha_{4,n_1,n_2} - \overline{\rho}_3\alpha_{2,n_1,n_2})\\
\frac{d\overline{\rho}_3}{dt} &= \overline{\rho}_3\left[-m + \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} \frac{\overline{\rho}_1^{n_1}\overline{\rho}_2^{n_2}}{n_1!n_2!}(c_1\alpha_{1,n_1,n_2} + c_2\alpha_{2,n_1,n_2})\right]
\end{aligned} \tag{A.14}$$

where

$$\alpha_{1,n_1,n_2} = \left(\frac{\partial^{n_1+n_2} f_1}{\partial\rho_1^{n_1}\partial\rho_2^{n_2}}\right)\Bigg|_{\overline{\rho}_1=\overline{\rho}_2=0}; \qquad \alpha_{2,n_1,n_2} = \left(\frac{\partial^{n_1+n_2} f_2}{\partial\rho_1^{n_1}\partial\rho_2^{n_2}}\right)\Bigg|_{\overline{\rho}_1=\overline{\rho}_2=0};$$

$$\alpha_{3,n_1,n_2} = \left(\frac{\partial^{n_1+n_2} g_1}{\partial\rho_1^{n_1}\partial\rho_2^{n_2}}\right)|_{\overline{\rho}_1=\overline{\rho}_2=0}; \qquad \alpha_{4,n_1,n_2} = \left(\frac{\partial^{n_1+n_2} g_2}{\partial\rho_1^{n_1}\partial\rho_2^{n_2}}\right)\Bigg|_{\overline{\rho}_1=\overline{\rho}_2=0};$$

Appendix B. The method of the simplest equation and its version based on the balance equation

There are many approaches for obtaining exact analytic solutions of nonlinear partial differential equations [26–30]. In this paper we use the modified method of the simplest equation. The schema of application of the modified method of the simplest equation is shown in Fig. B.4. The modified method of the simplest equation is a version of the method of the simplest equation [31,32] that is based on the fact that after application of an appropriate ansatz a large class of NPDEs can be reduced to ODEs of the kind ($\mathcal{P}$ means polynomial)

$$\mathcal{P}\left(F(\xi), \frac{dF}{d\xi}, \frac{d^2F}{d\xi^2}, \dots\right) = 0, \tag{B.1}$$

and for some equations of the kind (B.1) particular solutions can be obtained which are finite series

$$F(\xi) = \sum_{i=0}^{P} a_i[\Phi(\xi)]^i, \tag{B.2}$$

constructed by the solution $\Phi(\xi)$ of a simpler equation referred to as the simplest equation. The simplest equation can be the equation of Bernoulli, equation of Riccati, etc. The substitution of Eq. (B.2) in Eq. (B.1) leads to the polynomial equation

$$\mathcal{P} = \sigma_0 + \sigma_1\Phi + \sigma_2\Phi^2 + \cdots + \sigma_r\Phi^r = 0, \tag{B.3}$$

where the coefficients σ_i, $i = 0, 1, \dots, r$ depend on the parameters of the equation and on the parameters of the solutions. Equating all these coefficients to zero, i.e., by setting

$$\sigma_i = 0, \quad i = 1, 2, \dots, r, \tag{B.4}$$

one obtains a system of nonlinear algebraic equations. Each solution of this system leads to a solution of kind (B.2) of Eq. (B.1).

In order to obtain a non-trivial solution by the above method we have to ensure that σ_r contains at least two terms. To do this within the scope of the modified method of the simplest equation we have to balance the highest powers of Φ that are obtained from the different terms of the solved equation of kind (B.1). As a result of this we obtain an additional equation between some of the parameters of the equation and the solution. This equation is called the balance equation [33–37].

We note that the method of the simplest equation and its modified version are closely connected to the problem for obtaining meromorphic solutions of nonlinear partial differential equations [38,39]. By the methodology described in [38,39] one can obtain other interesting classes of solutions of nonlinear PDEs such as rational solutions for example. In addition we stress that by means of the traveling wave ansatz one reduces the nonlinear PDE to a nonlinear ODE and after this if an appropriate simplest ODE exists then a particular solution can be obtained that usually depends on as many parameters of the problem as possible. In many cases such particular solutions are among the few possible exact analytic solutions of the studied nonlinear PDE.

Appendix C. Maple program for solving the coupled waves case

Here we present a Maple program for obtaining the exact solution of a system of equations for description of coupled waves in a system of three populations discussed in Section 2.2. The program has the following parts.

Part 1: The equations

1. $> eq1 := -(v + D11) * diff(rho1(xi), xi) - D12 * diff(rho2(xi), xi) - D13 * diff(rho3(xi), xi) - r10 * rho1(xi) + r10 * alpha110 * rho1(xi)^2 - r10 * alpha120 * rho1(xi) * rho2(xi) - r10 * alpha130 * rho1(xi) * rho3(xi);$
2. $> eq2 := -D21 * diff(rho_1(xi), xi) - (v + D22) * diff(rho2(xi), xi) - D23 * diff(rho_3(xi), xi) - r20 * rho2(xi) + r20 * alpha210 * rho1(xi) * rho2(xi) - r20 * alpha220 * rho_2(xi)^2 - r20 * alpha230 * rho2(xi) * rho3(xi);$
3. $> eq3 := -D31 * diff(rho1(xi), xi) - D32 * diff(rho2(xi), xi) - (v + D33) * diff(rho3(xi), xi) - r30 * rho3(xi) + r30 * alpha310 * rho1(xi) * rho3(xi) - r30 * alpha320 * rho2(xi) * rho3(xi) - r30 * alpha330 * rho3(xi)^2;$

Part 2: Substitution of the relationships for $\rho_{1,2,3}$ in the equations

4. $> rho1(xi) := a0 + a1 * Phi(xi);$
5. $> rho2(xi) := b0 + b1 * Phi(xi);$
6. $> rho3(xi) := c0 + c1 * Phi(xi);$
7. $> eq1; eq2; eq3;$

Part 3: Substitution of the relationship for $\frac{d\Phi}{d\xi}$ in the equations

8. $> eq1a := subs(diff(Phi(xi), xi) = a * Phi(xi)^2 + b * Phi(xi) + c, eq1);$
9. $> eq2a := subs(diff(Phi(xi), xi) = a * Phi(xi)^2 + b * Phi(xi) + c, eq2);$
10. $> eq3a := subs(diff(Phi(xi), xi) = a * Phi(xi)^2 + b * Phi(xi) + c, eq3);$

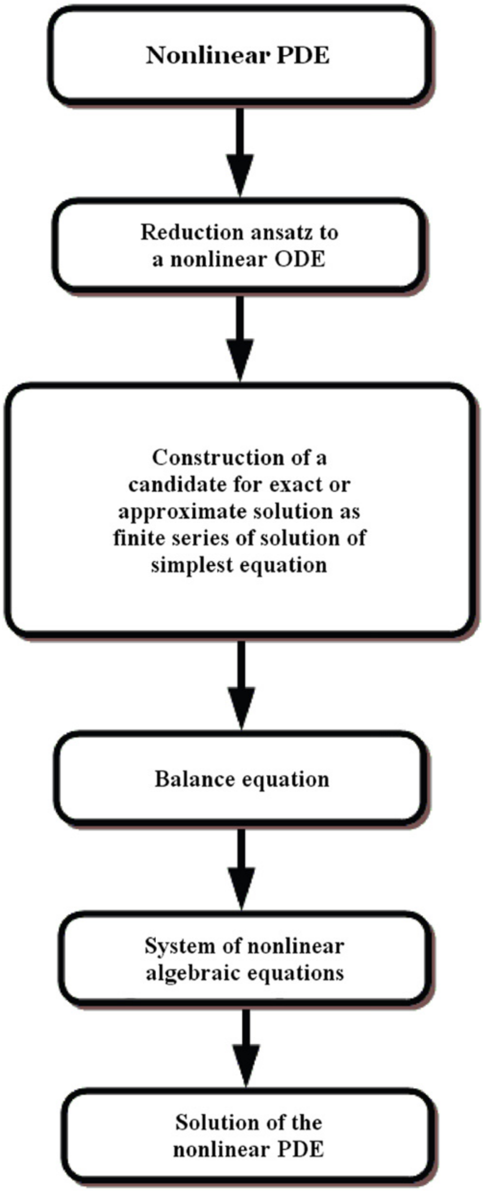

Fig. B.4. The modified method of the simplest equation.

Part 4: *Extracting the system of nonlinear algebraic equations*

11. $> eq1b := collect(eq1a, Phi(xi));$
12. $> eq2b := collect(eq2a, Phi(xi));$
13. $> eq3b := collect(eq3a, Phi(xi));$
14. $> e1 := coeff(eq1b, Phi(xi)^2);$
15. $> e2 := coeff(eq1b, Phi(xi));$
16. $> e3 := -(v + D11) * a1 * c - r10 * alpha120 * a0 * b0 - D13 * c1 * c - D12 * b1 * c - r10 * alpha130 * a0 * c0 - r10 * a0 + r10 * alpha110 * a0^2;$
17. $> e4 := coeff(eq2b, Phi(xi)^2);$
18. $> e5 := coeff(eq2b, Phi(xi));$
19. $> e6 := -D21 * a1 * c - r20 * alpha220 * b0^2 - D23 * c1 * c - (v + D22) * b1 * c - r20 * alpha230 * b0 * c0 - r20 * b0 + r20 * alpha210 * a0 * b0;$
20. $> e7 := coeff(eq3b, Phi(xi)^2);$
21. $> e8 := coeff(eq3b, Phi(xi));$

22. $> e9 := -D31 * a1 * c - r30 * alpha320 * b0 * c0 - (v + D33) * c1 * c - D32 * b1 * c - r30 * alpha330 * c0^2 - r30 * c0 + r30 * alpha310 * a0 * c0;$

Part 5: *Simplifying assumptions*

23. $> r10 := 1; r20 := 1; r30 := 1; alpha110 := 1; alpha220 := 1; alpha330 := 1; alpha120 := 1; alpha130 := 1; alpha210 := 1; alpha230 := 1; alpha310 := 1; alpha320 := 1; D21 := D12; D31 := D13; D32 := D23; D22 := D11; D33 := D11; D12 := 1; D13 := 1; D23 := 1;$
24. $> e1; e2; e3; e4; e5; e6; e7; e8; e9;$

Part 6: *Solution of the system of nonlinear algebraic equations*

25. $> sol1 := solve(e1, c_1);$
26. $> c1 := sol1;$
27. $> sol2 := solve(e2, v);$
28. $> v := sol2;$
29. $> sol3 := solve(e3, c);$
30. $> c := sol3;$
31. $> sol4 := solve(e4, a);$
32. $> a := sol4[1];$
33. $> sol5 := solve(e5, a_0);$
34. $> a_0 := sol5;$
35. $> sol6 := solve(e8, b_0);$
36. $> b0 := sol6;$

By this program one obtains relationships for the parameters a_0, b_0, c_1, a, c, v.

Appendix D. Fluctuations, diffusion Markov process and Fokker–Planck equation

In the paper we discuss equations of the kind

$$\frac{dx}{dt} = X[x(t)] + \eta(t) \tag{D.1}$$

where the process $\eta(t)$ models small fluctuations. Let $\eta(t) = \sigma\xi(t)$ where $\sigma > 0$ is the intensity factor and the covariance function of the process $\xi(t)$ be a δ-function $E[\xi(t)\xi'(t)] = \delta(t - t')$. If in addition the expected value of $\xi(t)$ is zero: $E[\xi(t)] = 0$ the process $\xi(t)$ is called white noise and the equation

$$\frac{dx}{dt} = X[x(t)] + \sigma\xi(t); \qquad x(0) = x_0, \tag{D.2}$$

is called the Langevin equation.

$\xi(t)$ can be written as a time derivative of a Wiener process W_t (for this process $W_0 = 0$; the function W_t is almost surely continuous; and W_t has independent increments $W_t - W_s(0 \le s < t)$ which are normally distributed with expected value 0 and variance equal to $t - s$):

$$\xi(t) = \frac{dW_t}{dt} \;\rightarrow W_t = \int_0 ds\, \xi(s) \tag{D.3}$$

Then the Langevin equation can be written in the form

$$dx_t = X(x_t)dt + \sigma dW_t; \quad x_0 : \text{random}. \tag{D.4}$$

After one integration of the Langevin equation one obtains

$$x_t(\omega) = x_0(\omega) + \int_0 dsX[x_s(\omega)] + \sigma W_t(\omega). \tag{D.5}$$

As the white noise is δ-correlated the solution of Eq. (D.5) is a homogeneous Markov process. The infinitesimal generator A of the solution process x_t [20,40] is a differential operator of second order

$$(Ag)(x) = f(x)g'(x) + \frac{\sigma^2}{2}g''(x). \tag{D.6}$$

The form of the infinitesimal operator A is important as it is determines the form of the Fokker–Planck equation for the one dimensional distribution $p(x, t)$ connected to the solution process (D.5). This Fokker–Planck equation is

$$\frac{\partial p(x,t)}{\partial t} = -\frac{\partial}{\partial x}[X(x)p(x,t)] + \frac{\sigma^2}{2}\frac{\partial^2 p(x,t)}{\partial x^2}; \qquad p(x,0) = p_0(x). \tag{D.7}$$

We note that if we set $\sigma^2 = b$ we obtain Eq. (3.2) from the main text of the paper. By setting

$$J_p = pX - \frac{\sigma^2}{2}\frac{\partial p}{\partial x} \tag{D.8}$$

we can write Eq. (D.7) in the form

$$\frac{\partial p}{\partial t} + \frac{\partial J_p}{\partial x} = 0 \tag{D.9}$$

Eq. (D.9) is a balance equation for the probability. There is no production of probability and the balance equation shows that the transport of probability along x happens by 'flow', determined by the 'velocity' field X and 'diffusion' determined by $-\frac{\sigma^2}{2}\frac{\partial p}{\partial x}$. This 'diffusion' tries to decrease the differences in the probability distribution.

Because of the above the considered class of Markov processes are called diffusion processes. X is the drift of the diffusion process and σ^2 is called the diffusion coefficient of the diffusion process.

We finish this short discussion by two observations that are important for the text in the body of the paper [20]

Observation 1. *In the stationary case $p(x,t) \to p(x,0) = p_0(x)$ the stochastic system described by Eq. (D.4) spends much time around the stable fixed points of the corresponding deterministic system*

$$\frac{dx}{dt} = X(x)$$

Observation 1 is a consequence of the fact that for the stationary case the Fokker–Planck equation (D.7) becomes

$$0 = -\frac{d}{dx}[X(x)p_0(x)] + \frac{\sigma^2}{2}\frac{d^2 p_0(x)}{dx^2}. \tag{D.10}$$

After one integration one obtains

$$J_p^0 = -X(x)p_0(x) + \frac{\sigma^2}{2}\frac{dp_0(x)}{dx} = \text{const}. \tag{D.11}$$

The integration constant is equal to zero because of the boundary condition $J_p^0 = 0|_{\pm\infty}$ and for $p_0(x)$ one obtains

$$p_0(x) = \frac{\exp\left[-\frac{2}{\sigma^2}V(x)\right]}{\int_{-\infty}^{\infty} dx\, \exp\left[-\frac{2}{\sigma^2}V(x)\right]} \tag{D.12}$$

where

$$V(x) = -\int^x dz\, X(z) \tag{D.13}$$

The local extrema of $p_0(x)$ are given by

$$\left.\frac{dp_0}{dx}\right|_{x_0} = 0 \xrightarrow{(D.12)} \left.\frac{dV}{dx}\right|_{x_0} = 0 \xrightarrow{(D.13)} f(x_0) = 0. \tag{D.14}$$

Thus the extrema of $p_0(x)$ coincide with the fixed points of the corresponding deterministic differential equation. The maxima of $p_0(x)$ are located at the stable fixed points and the minima of $p_0(x)$ are located at the unstable fixed points.

The second observation is connected to the importance of the stationary distribution $p_0(x)$:

Observation 2. *Each time dependent solution $p(x,t)$ of the Fokker–Planck equation (D.7) converges at $t \to \infty$ to the stationary distribution $p_0(x)$ from Eq. (D.12) (if $p_0(x)$ exists).*

In order to show that Observation 2 is true we shall use the technique of the Lyapunov functional $H(t)$. Let us discuss the functional

$$H(t) = \int_{-\infty}^{\infty} dx\, p(x,t) \ln \frac{p(x,t)}{p_0(x)} \tag{D.15}$$

where $p(x,t)$ is an arbitrary solution of the Fokker–Planck equation (D.7). Using the normalization $\int_{-\infty}^{\infty} dx\, p(x,t) = 1$ for each $t > 0$ and the fact that $\ln(1/y) \geq 1 - y$ for $y > 0$ one can write the inequality

$$H(t) = \int_{-\infty}^{\infty} dx\, p(x,t)\left[\ln\frac{p(x,t)}{p_0(x)} + \frac{p_0(x)}{p(x,t)} - 1\right] \geq 0 \tag{D.16}$$

where the equality arises for $p(x,t) = p_0(x)$. We shall show that $dH/dt \leq 0$ for each t. We take the derivative of H with respect to t and by means of the Fokker–Planck equation (D.7) after some calculations we obtain the relationships

$$\frac{dH}{dt} = \int_{-\infty}^{\infty} dx \left[-\frac{\partial}{\partial x}[X(x)p(x,t)] + \frac{\sigma^2}{2}\frac{\partial^2 p(x,t)}{\partial x^2}\right]\left[\ln\frac{p(x,t)}{p_0(x)}\right]$$
$$= -\frac{\sigma^2}{2}\int_{-\infty}^{\infty} dx p(x,t)\left[\frac{p_0(x)}{p(x,t)}\frac{\partial}{\partial x}\left(\frac{p_0(x)}{p(x,t)}\right)\right]^2. \tag{D.17}$$

The last integral in (D.17) is positive for $p \neq p_0$ and it is equal to zero only when $p(x,t) = p_0(x)$. Then $\frac{dH}{dt} \leq 0$ and $\frac{dH}{dt} = 0$ only when $p(x,t) = p_0(x)$. From (D.16) and from $\frac{dH}{dt} \leq 0$ it follows that

$$\lim_{t\to\infty} p(x,t) = p_0(x) \tag{D.18}$$

which is exactly the essence of Observation 2.

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